1. Motivation

The following intrinsic property of functions guarantees whether left or right inverses exist:

**Theorem 1.** If \( f \circ h = 1_B \) then \( f \) is a surjection and \( h \) is an injection.

This result is well known and is proved in [1]. Relations generalize functions. Do similar properties exist for relations? In the following, we show the following new result:

**Theorem 2.** Given relations \( R \) and \( S \) and function \( f \):

1. If \( R \circ S = 1_B \) then \( R \) is a right total relation and \( S \) is a left total relation.
2. If \( f \circ S = 1_B \) then \( f \) is a surjection and \( S \) is both a left unique and left total relation.

In general, the converses do not hold. The proof is presented in Section 3 after reviewing some preliminary definitions and properties.

2. Properties of Relations

Sets are denoted by letters \( A, B, C, \ldots \). The sets we consider are finite and are represented explicitly by elements \( A = \{a_1, a_2, \ldots, a_n\} \) and \( B = \{b_1, b_2, \ldots, b_k\} \). Denote the cardinality of a set \( A \) by \( |A| \); thus \( |A| = n \) and \( |B| = k \).

A relation \( R \) between two sets \( A \) and \( B \) is any subset of the Cartesian product \( A \times B \): \( R \subseteq A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\} \). Relations are denoted by capital letters \( R, S, T, U, V, \ldots \).

The opposite relation \( R^o \) is defined by switching the order of the pairs in \( R \). So \( R^o : B \to A \) with \( R^o = \{(b,a) \mid (a,b) \in R\} \).

Abbreviate the \( R \subseteq A \times B \) by the relation specification \( R : A \to B \). As in the case with function composition, a relation \( R \) between sets \( A \) and \( B \) can be combined with another relation \( S \) between sets \( B \) and \( C \). This relation creates a third relation \( S \circ R : A \to C \) between sets \( A \) and \( C \) in the following way: \( S \circ R = \{(a,c) \mid \text{for any } b \in B, (a,b) \in R \text{ and } (b,c) \in S\} \). Relation composition is associative: \( R \circ (S \circ T) = (R \circ S) \circ T \).
The identity relation on a set is defined by \( 1_A : A \to A \) with \( 1_A = \{ (a, a) \mid a \in A \} \). The identity relation has the property that \( 1_B \circ R = R \circ 1_A = R \). An inverse relation reverses the effects of the relation. Suppose \( R \circ S = 1_B \) and \( T \circ R = 1_A \): relation \( S \) is called a right inverse and relation \( T \) is called a left inverse. If the inverses both simultaneously exist then they are equal: associativity of relation composition implies: \( T \circ 1_B = T \circ (R \circ S) = (T \circ R) \circ S = 1_A \circ S = S \).

A relation \( R \) is right-unique if for \( a \in A \) and \( b_1, b_2 \in B \), if \( (a, b_1) \in R \) and \( (a, b_2) \in R \) then \( b_1 = b_2 \). Right unique relations are called simple by Hoo[p.25] and Bird DeMoor[p. 88]. A relation \( R \) is right-unique if \( R \circ R^o \subseteq 1_B \). Denote the set of all right unique relations between sets \( A \) and \( B \) by \( RU \). A relation \( R \) is right-total if for each \( b \in B \) there exists an \( a \in A \) such that \( (a, b) \in R \). Right total relations are called total by Hoogendijk [2, p.25] and entire by Bird & De Moor[3, p. 88]. A relation \( R \) is right-total if \( R \circ R^o \supseteq 1_B \). Denote the set of all right total relations between sets \( A \) and \( B \) by \( RT \).

A relation \( R \) is left-total if \( R^o \) is right-total. A relation \( R \) is left-total if \( R^o \circ R \supseteq 1_A \). Denote the set of all left total relations between sets \( A \) and \( B \) by \( LT \). A relation \( R \) is left-unique if for \( a_1, a_2, \in A \) and \( b \in B \), if \( (a_1, b) \in R \) and \( (a_2, b) \in R \) then \( a_1 = a_2 \). A relation \( R \) is left-unique if \( R \circ R^o \subseteq 1_A \). Denote the set of all left unique relations between sets \( A \) and \( B \) by \( LU \).

A function is a relation that is left-total and right-unique. Functions are denoted by lower case letters \( f, g, h, \cdots \). The set of all functions is \( RU \cap LT \). A surjection is a right-total function: the set of all surjections is \( RU \cap LT \cap RT \). An injection is a left-unique function: the set of all injections is \( RU \cap LT \cap LU \). A bijection is a function that is an injection and a surjection: the set of all bijections is \( RU \cap LT \cap LU \cap RT \). Note that if \( f \) and \( f^o \) are both functions then \( f \) and \( f^o \) must be bijections. Note that the identity relation is also a bijection.

The following basic rules are proved by Hoogendijk [2, p. 24] and Bird & de Moor, [3, p.90]:

**Lemma (Basic Relation Rules).**

1. \( R \subseteq S \) if and only if \( R^o \subseteq S^o \).
2. \( (R \circ S)^o = S^o \circ R^o \).
3. \( R \subseteq R \circ R^o \circ R \).
4. (Shunt) \( f \circ U \subseteq V \) if and only if \( U \subseteq f^o \circ V \) and \( U \circ g^o \subseteq V \) if and only if \( U \subseteq V \circ g \).
3. Proof of New Results

Theorem 2. Given relations $R$ and $S$ and function $f$:

(1) If $R \circ S = 1_B$ then $R \in RT$ and $S \in LT$.

(2) If $f \circ S = 1_B$ then $f$ is a surjection and $S \in LU \cap LT$.

Proof of (1). Rule 3 in the Lemma, $R \subseteq R \circ R^o \circ R$ implies $R \circ S \subseteq (R \circ R^o \circ R) \circ S$. Applying the condition (1) in the Theorem implies $1_B \subseteq R \circ R^o$. Thus $R \in RT$. Also note that $R \circ S \subseteq R \circ S \circ S^o \circ S$ so $1_B \subseteq S^o \circ S$. Thus $S \in LT$. QED.

Note that the converse is not true. For example, if $R^* = \{(a_1,b_1),(a_1,b_2),(a_2,b_1)\}$ and $S^* = \{(b_1,a_1),(b_2,a_2)\}$, then $R^*$ is right total and $S^*$ is left total. However, $R^* \circ S^* = \{(b_1,b_1),(b_1,b_2),(b_2,b_1)\} \neq 1_B$.

Proof of (2). First, note if $f \circ S = 1_B$ then $f \circ S \subseteq 1_B$. Now use Rule 4 (the “Shunt Rule”) in the Lemma. This implies for $(U = S$ and $V = 1_B)$ $S \subseteq f^o$. Composition on both sides with $f$ implies $f \circ S = 1_B \subseteq f \circ f^o$. Thus $f \in RT$ so $f$ is a surjection. Rule 2 implies $S^o \circ S \subseteq S^o \circ f^o = (f \circ S)^o = 1_B$, so $S \in LU$. Now use the first part (1) of the condition so $S \in LT$ as well. We have $S \in LU \cap LT$. QED.

Note that the (2) implies that if $S$ is also right unique then $S \in RU \cap LU \cap LT$. This shows that $S$ is an injection, so (2) also proves Theorem 1.

4. References

